

On the Gabor frame set for compactly supported continuous functions

Ole Christensen*, Hong Oh Kim†, Rae Young Kim‡

February 19, 2016

Abstract

We identify a class of continuous compactly supported functions for which the known part of the Gabor frame set can be extended. At least for functions with support on an interval of length two, the curve determining the set touches the known obstructions. Easy verifiable sufficient conditions for a function to belong to the class are derived, and it is shown that the B-splines B_N , $N \geq 2$, as well as certain “continuous and truncated” versions of several classical functions (e.g., the Gaussian and the two-sided exponential function) belong to the class. The sufficient conditions for the frame property guarantees the existence of a dual window with a prescribed size of the support.

Keywords: Gabor frames, Frame set, B-splines

2010 Mathematics Subject Classification: 42C15, 42C40

1 Introduction

Frames is a functional analytic tool to obtain representations of the elements in a Hilbert space as a (typically infinite) superposition of building blocks. Frames indeed lead to decompositions that are similar to the ones obtained via orthonormal bases, but with much greater flexibility, due to the fact that the definition is significantly less restrictive. For example, in contrast

*Department of Applied Mathematics and Computer Science, Technical University of Denmark, Building 303, 2800 Lyngby, Denmark (ochr@dtu.dk)

†Division of General Studies, UNIST, 50 UNIST-gil, Ulsan 44919, Republic of Korea (hkim2031@unist.ac.kr)

‡Department of Mathematics, Yeungnam University, 280 Daehak-Ro, Gyeongsan, Gyeongbuk 38541, Republic of Korea (rykim@ynu.ac.kr)

to the case for a basis, the elements in a frame are not necessarily (linearly) independent, i.e., frames can be redundant.

One of the main manifestations of frame theory is within Gabor analysis, where the aim is to obtain efficient representations of signals in a way that reflects the time-frequency distribution. For any $a, b > 0$, consider the translation operator T_a and the modulation operator E_b , both acting on the particular Hilbert space $L^2(\mathbb{R})$, given by $T_a f(x) = f(x - a)$, respectively $E_b f(x) = e^{2\pi i b x} f(x)$. Given $g \in L^2(\mathbb{R})$, the collection of functions $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is called a (*Gabor*) *frame* if there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 \leq B \|f\|^2, \forall f \in L^2(\mathbb{R}).$$

If at least the upper condition is satisfied, $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is called a *Bessel sequence*. It is known that for every frame $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ there exists a dual frame $\{E_{mb} T_{na} h\}_{m,n \in \mathbb{Z}}$ such that each $f \in L^2(\mathbb{R})$ has the decomposition

$$f = \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb} T_{na} h \rangle E_{mb} T_{na} h. \quad (1.1)$$

The problem of determining $g \in L^2(\mathbb{R})$ and parameters $a, b > 0$ such that $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a frame has attracted a lot of attention over the past 25 years. The *frame set* for a function $g \in L^2(\mathbb{R})$ is defined as the set

$$\mathcal{F}_g := \{(a, b) \in \mathbb{R}_+^2 \mid \{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}} \text{ is a frame for } L^2(\mathbb{R})\}.$$

Clearly the “size” of the set \mathcal{F}_g reflects the flexibility of the function g in regard of obtaining expansions of the type (1.1). In particular it is known that $ab \leq 1$ is necessary for $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ to be a frame and that the number $(ab)^{-1}$ is a measure of the redundancy of the frame; the smaller the number is, the more redundant the frame will be. Thus a reasonable function g should lead to a frame $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ for values $(ab)^{-1}$ that are reasonably close to one. We remark that \mathcal{F}_g is known to be open if g belongs to the Feichtinger algebra; see [9, 1].

Until recently the exact frame set was only known for very few functions: the Gaussian $g(x) = e^{-x^2}$ [22, 26, 27], the hyperbolic secant [17], and the functions $h(x) = e^{-|x|}$, $k(x) = e^{-x} \chi_{[0, \infty[}(x)$ [13, 16]. In [12] a characterization was obtained for the class of totally positive functions of finite type, and based on [15] the frame set for functions $\chi_{[0, c]}$, $c > 0$, was characterized in [7].

For the sake of applications of Gabor frames it is essential that the window g is a continuous function with compact support. Most of the related literature deals with special types of functions like truncated trigonometric functions or various types of splines, see [8, 20, 19, 18]. Various classes of functions have also been considered, e.g., functions yielding a partition of unity [11, 5], functions with short support or a finite number of sign-changes [3, 4, 6], or functions that are bounded away from zero on a specified part of the support [21]. The case of B-spline generated Gabor systems has attracted special attention, see, e.g., [24, 19, 6, 21, 10].

To our best knowledge the frame set has not been characterized for any function $g \in C_c(\mathbb{R}) \setminus \{0\}$. We will, among others, consider a class of functions for which we can extend the known set of parameters (a, b) yielding a Gabor frame. The class of functions contains the B-splines B_N , $N \geq 2$, as well as certain “continuous and compactly supported variants” of the above functions g, h and other classical functions. Furthermore, the results guarantees the existence of dual windows with a support size given in terms of the translation parameter.

In the rest of this introduction we will describe the relevant class of windows and their frame properties. Proofs of the frame properties are in Section 2, and easy verifiable conditions for a function to belong to the class are derived in Section 3.

Let us first collect some of the known results concerning frame properties for continuous compactly supported functions; (i) is classical, and we refer to [2] for a proof.

Proposition 1.1 *Let $N > 0$, and assume that $g : \mathbb{R} \rightarrow \mathbb{C}$ is a continuous function with $\text{supp } g \subseteq [-\frac{N}{2}, \frac{N}{2}]$. Then the following holds:*

- (i) *If $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame, then $ab < 1$ and $a < N$.*
- (ii) [21] *Assume that $0 < a < N, 0 < b \leq \frac{2}{N+a}$, and $\inf_{x \in [-\frac{a}{2}, \frac{a}{2}]} |g(x)| > 0$. Then $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame, and there is a unique dual window $h \in L^2(\mathbb{R})$ such that $\text{supp } h = [-\frac{a}{2}, \frac{a}{2}]$.*
- (iii) [6] *Assume that $\frac{N}{2} \leq a < N$ and $0 < b < \frac{1}{a}$. If $g(x) > 0$, $x \in]-\frac{N}{2}, \frac{N}{2}[$, then $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame.*

We will now introduce the window class that will be used in the current paper; it is a subset of the set of functions g considered in Proposition 1.1 (iii). The definition is inspired by certain explicit estimates for B-splines, given by Trebels and Steidl in [28]; this point will be clear in Proposition

3.1. First, fix $N > 0$ and $0 < a < N$. Consider the first order difference $\Delta_a f$ and the second order difference $\Delta_a^2 f$, given by

$$\Delta_a f(x) = f(x) - f(x - a), \quad \Delta_a^2 f(x) = f(x) - 2f(x - a) + f(x - 2a).$$

We define the window class as the set of functions

$$V_{N,a} := \{f \in C(\mathbb{R}) \mid \text{supp } f = [-\frac{N}{2}, \frac{N}{2}], f \text{ is real-valued and satisfies (A1)-(A3)}\},$$

where

(A1) f is symmetric around the origin;

(A2) f is strictly increasing on $[-\frac{N}{2}, 0]$;

(A3) If $a < \frac{N}{3}$, then $\Delta_a^2 f(x) \geq 0, x \in [-\frac{N}{2}, -\frac{N}{4} + \frac{3a}{4}]$; if $a \geq \frac{N}{3}$, then $\Delta_a^2 f(x) \geq 0, x \in [-\frac{N}{2}, 0] \cup \{-\frac{N}{4} + \frac{3a}{4}\}$.

Note that by the symmetry condition (A1) a function $f \in V_{N,a}$ is completely determined by its behavior for $x \in [-\frac{N}{2}, 0]$. If $a \geq \frac{N}{3}$, the point $-\frac{N}{4} + \frac{3a}{4}$ considered in (A3) is not contained in $[-\frac{N}{2}, 0]$; however, if desired, the symmetry condition allows to formulate the condition $\Delta_a^2 f(-\frac{N}{4} + \frac{3a}{4}) \geq 0$ alternatively as

$$f(\frac{N}{4} - \frac{3a}{4}) - 2f(-\frac{N}{4} - \frac{a}{4}) \geq 0 \quad (1.2)$$

because the argument $x - 2a$ of the last term in the second order difference is less than $-\frac{N}{2}$.

The definition of $V_{N,a}$ is technical, but we will derive easy verifiable conditions for a function g to belong to this set in Proposition 3.1, and also provide several natural examples of such functions. Our main result extends the range of $b > 0$ yielding a frame, compared with Proposition 1.1 (ii):

Theorem 1.2 *For $N > 0$, let $0 < a < N$ and $\frac{2}{N+a} < b \leq \frac{4}{N+3a}$. Assume that $g \in V_{N,a}$. Then the Gabor system $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, and there is a unique dual window $h \in L^2(\mathbb{R})$ such that $\text{supp } h \subseteq [-\frac{3a}{2}, \frac{3a}{2}]$.*

Membership of a function g in a set $V_{N,a}$ for some $a \in]0, N[$ only gives information about the frame properties of $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ for this specific value of the translation parameter a . In order to get an impression of the frame properties of $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ in a region in the (a, b) -plane, we need to consider a function g that belongs to $V_{N,a}$ for an interval of a -values, preferably for all $a \in]0, N[$. Fortunately several natural functions have this property. The following list collects some of the results we will obtain in Section 3. Considering any $N \in \mathbb{N} \setminus \{1\}$,

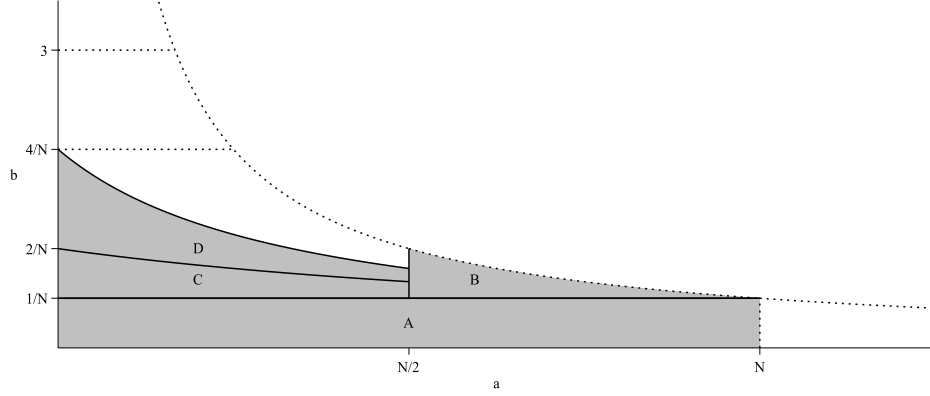


Figure 1: The Figure shows the following regions: B as in Proposition 1.1 (iii), and D as in Theorem 1.2. The region A corresponds to the case where the frame operator is a multiplication operator, and $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame if $\inf_{x \in [0,a]} \sum_{n \in \mathbb{Z}} |g(x-na)|^2 > 0$. The region C is a part of the region determined by Proposition 1.1 (ii), corresponding to the new findings in the paper [21].

- The B-spline B_N of order N belongs to $\bigcap_{0 < a < N} V_{N,a}$;
- The function $f_N(x) := \cos^{2N-2}(\frac{\pi x}{N}) \chi_{[-\frac{N}{2}, \frac{N}{2}]}(x)$ belongs to $\bigcap_{0 < a < N} V_{N,a}$;
- The function $h_N(x) := \left(e^{-|x|} - e^{-\frac{N}{2}}\right) \chi_{[-\frac{N}{2}, \frac{N}{2}]}(x)$ belongs to $\bigcap_{0 < a < N} V_{N,a}$;
- The function $g_N(x) := \left(e^{-x^2} - e^{-\frac{N^2}{4}}\right) \chi_{[-\frac{N}{2}, \frac{N}{2}]}(x)$ belongs to $\bigcap_{\frac{3N}{7} \leq a < N} V_{N,a}$.

In particular, Proposition 1.1 and Theorem 1.2 imply that for $N \in \mathbb{N} \setminus \{1\}$ the functions B_N , f_N , and h_N generate frames whenever $0 < a < N$ and $0 < b \leq \frac{4}{N+3a}$; and g_N generates a frame whenever $\frac{3N}{7} \leq a < N$ and $0 < b \leq \frac{4}{N+3a}$.

Note that the limit curve $b = \frac{4}{N+3a}$ in Theorem 1.2 touches the known obstructions for Gabor frames. In fact, for $N = 2$ we obtain that $b \rightarrow 2$ whenever $a \rightarrow 0$. Since it is known that the B-spline B_2 does not generate a frame for $b = 2$ [8, 11] we can not go beyond this. We also know that at least for some functions $g \in \bigcap_{0 < a < N} V_{N,a}$ parts of the region determined by the inequalities $b < 2, a < 2, ab < 1$ do not belong to the frame set. Considering for example the B-spline B_2 , [21] shows that the point $(a, b) =$

$(\frac{2}{7}, \frac{7}{4})$ does not belong to the frame set. For $a = \frac{2}{7}$ Theorem 1.2 guarantees the frame property for $b < \frac{7}{5}$, which is close to the obstruction. These considerations indicate that the frame region in Theorem 1.2 in a quite accurate way describes the maximally possible frame set below $b = 2$ that is valid for all the functions in $V_{N,a}$, at least for $N = 2$.

2 Frame properties for functions $g \in V_{N,a}$

The purpose of this section is to prove Theorem 1.2. Since the functions $g \in V_{N,a}$ are bounded and have compact support, they generate Bessel sequences $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ for all $a, b > 0$. By the duality conditions [25, 14], two bounded functions g, h with compact support generate dual frames $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n \in \mathbb{Z}}$ for some fixed $a, b > 0$ if and only if

$$\sum_{m \in \mathbb{Z}} g(x - \ell/b + ma) \overline{h(x + ma)} = b\delta_{\ell,0}, \text{ a.e. } x \in [-\frac{a}{2}, \frac{a}{2}];$$

in particular, a function $g \in V_{N,a}$ and a bounded real-valued function h with support on $[-\frac{3a}{2}, \frac{3a}{2}]$ generate dual Gabor frames $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n \in \mathbb{Z}}$ for $L^2(\mathbb{R})$ for some $b \leq \frac{4}{N+3a}$ if and only if the equations

$$\sum_{m=-1}^1 g(x - \ell/b + ma) h(x + ma) = b\delta_{\ell,0}, \text{ a.e. } x \in [-\frac{a}{2}, \frac{a}{2}]. \quad (2.1)$$

hold for $\ell = 0, \pm 1$. Given $g \in V_{N,a}$ we will therefore consider the 3×3 matrix-valued function G on $[-\frac{a}{2}, \frac{a}{2}]$ defined by

$$G(x) := (g(x - \frac{\ell}{b} + ma))_{-1 \leq \ell, m \leq 1} = \begin{pmatrix} g(x + \frac{1}{b} - a) & g(x + \frac{1}{b}) & g(x + \frac{1}{b} + a) \\ g(x - a) & g(x) & g(x + a) \\ g(x - \frac{1}{b} - a) & g(x - \frac{1}{b}) & g(x - \frac{1}{b} + a) \end{pmatrix}.$$

In terms of the $G(x)$ the condition (2.1) simply means that

$$G(x) \begin{pmatrix} h(x - a) \\ h(x) \\ h(x + a) \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \text{ a.e. } x \in [-\frac{a}{2}, \frac{a}{2}]. \quad (2.2)$$

We will show that the matrix $G(x)$ is invertible for all $x \in [-\frac{a}{2}, \frac{a}{2}]$; this will ultimately give us a bounded and compactly supported function h satisfying (2.1) and hereby prove Theorem 1.2. The invertibility of $G(x)$ will be derived as a consequence of a series of lemmas, where we first consider $x \in [-\frac{a}{2}, 0]$. Note that the proof of the first result does not use the property (A3):

Lemma 2.1 For $N > 0$, let $0 < a < N$ and $\frac{2}{N+a} < b \leq \frac{4}{N+3a}$. Assume that $g \in V_{N,a}$, and let $x \in [-\frac{a}{2}, 0]$. Then the following hold:

- (a) $g(x + \frac{1}{b} + a) \leq g(x + \frac{1}{b}) < g(x + \frac{1}{b} - a) \neq 0$;
- (b) $g(x - \frac{1}{b} - a) \leq g(x - \frac{1}{b}) < g(x - \frac{1}{b} + a) \neq 0$;
- (c) $g(x) > g(x - a)$ and $g(x) \geq g(x + a)$, with equality only for $x = -\frac{a}{2}$.

Proof. For (a), let $x \in [-\frac{a}{2}, 0]$. Using $b \leq \frac{4}{N+3a}$ and $a < N$,

$$x + \frac{1}{b} - a \geq \frac{1}{b} - \frac{3a}{2} \geq \frac{N+3a}{4} - \frac{3a}{2} > -\frac{N}{2}.$$

It follows that

$$x + \frac{1}{b} - a \subseteq [\frac{1}{b} - \frac{3a}{2}, \frac{1}{b} - a] \subseteq [-\frac{N}{2}, \frac{N}{2}]. \quad (2.3)$$

Using that $b \leq \frac{4}{N+3a} < \frac{1}{a}$ it follows that $-2x - \frac{2}{b} + a \leq 2(a - \frac{1}{b}) < 0$; thus

$$-x - \frac{1}{b} + a < x + \frac{1}{b}. \quad (2.4)$$

Since $g(x) > 0$ for $x \in]-\frac{N}{2}, \frac{N}{2}[$, we have by (2.3) that $g(x + \frac{1}{b} - a) \neq 0$. By (A2) we know that g is strictly decreasing on $[0, \frac{N}{2}]$. If $x + \frac{1}{b} - a \leq 0$, then we have by (2.3), (2.4) and the symmetry of g that

$$g(x + \frac{1}{b} - a) = g(-x - \frac{1}{b} + a) > g(x + \frac{1}{b}) \geq g(x + \frac{1}{b} + a);$$

if $x + \frac{1}{b} - a > 0$, then we have $g(x + \frac{1}{b} - a) > g(x + \frac{1}{b}) \geq g(x + \frac{1}{b} + a)$. Hence (a) holds. Similarly, (b) and (c) hold. \square

We now show that if $g \in V_{N,a}$ and $a \geq N/3$, the condition (A3) automatically holds on a larger interval.

Lemma 2.2 For $N > 0$, let $0 < a < N$. Assume that $g \in V_{N,a}$. Then $\Delta_a^2 g(x) \geq 0$, $\forall x \in [-\frac{N}{2}, -\frac{N}{4} + \frac{3a}{4}]$.

Proof. It suffices to show that for $a \geq \frac{N}{3}$,

$$\Delta_a^2 g(x) \geq 0, \forall x \in [0, -\frac{N}{4} + \frac{3a}{4}].$$

We first note that (A1) and (A2) imply that for $x \in [0, -\frac{N}{4} + \frac{3a}{4}]$,

$$g(x) \geq g(-\frac{N}{4} + \frac{3a}{4}) \text{ and } g(x-a) \leq g(-\frac{N}{4} - \frac{a}{4}). \quad (2.5)$$

For $a \geq \frac{N}{3}$, we have $-\frac{N}{4} - \frac{5a}{4} < -\frac{N}{2}$; due to the compact support of g , $f(-\frac{N}{4} - \frac{5a}{4}) = 0$; thus using (A1) again,

$$\begin{aligned} \Delta_a^2 g(-\frac{N}{4} + \frac{3a}{4}) &= g(-\frac{N}{4} + \frac{3a}{4}) - 2g(-\frac{N}{4} - \frac{a}{4}) \\ &= g(\frac{N}{4} - \frac{3a}{4}) - 2g(-\frac{N}{4} - \frac{a}{4}). \end{aligned}$$

Together with (2.5) this shows that

$$0 \leq \Delta_a^2 g(-\frac{N}{4} + \frac{3a}{4}) \leq \Delta_a^2 g(x), \quad \forall x \in [0, -\frac{N}{4} + \frac{3a}{4}],$$

as desired. \square

Let $G_{ij}(x)$ denote the ij -th minor of $G(x)$, the determinant of the submatrix obtained by removing the i -th row and the j -th column from $G(x)$.

Lemma 2.3 *For $N > 0$, let $0 < a < N$ and $\frac{2}{N+a} < b \leq \frac{4}{N+3a}$. Assume that $g \in V_{N,a}$ and let $x \in [-\frac{a}{2}, 0]$. Then the following hold:*

- (a) $G_{21}(x) \geq 0$, and equality holds iff $g(x + \frac{1}{b}) = g(x + \frac{1}{b} + a) = 0$;
- (b) $G_{23}(x) \geq 0$, and equality holds iff $g(x - \frac{1}{b} - a) = g(x - \frac{1}{b}) = 0$;
- (c) $G_{22}(x) \geq G_{21}(x) + G_{23}(x)$.

Proof. Since $g \geq 0$, (a) and (b) follow from Lemma 2.1 (a) & (b). For (c), we note that $\Delta_a g(x) = g(x) - g(x-a) = g(-x) - g(-x+a) = -\Delta_a g(-x+a)$ by the symmetry of g . Now a direct calculation shows that

$$\begin{aligned} &G_{22}(x) - G_{21}(x) - G_{23}(x) \\ &= -\Delta_a g(x + \frac{1}{b}) \Delta_a g(x - \frac{1}{b} + a) + \Delta_a g(x + \frac{1}{b} + a) \Delta_a g(x - \frac{1}{b}) \\ &= \Delta_a g(-x - \frac{1}{b} + a) \Delta_a g(x - \frac{1}{b} + a) - \Delta_a g(-x - \frac{1}{b}) \Delta_a g(x - \frac{1}{b}). \end{aligned}$$

Hence it suffices to show that

$$\Delta_a g(x - \frac{1}{b} + a) \geq \Delta_a g(x - \frac{1}{b}), \quad x \in [-\frac{a}{2}, 0]$$

and

$$\Delta_a g(-x - \frac{1}{b} + a) \geq \Delta_a g(-x - \frac{1}{b}), \quad x \in [-\frac{a}{2}, 0],$$

or, equivalently, that $\Delta_a^2 g(x - \frac{1}{b} + a) \geq 0$ and $\Delta_a^2 g(-x - \frac{1}{b} + a) \geq 0$, both for $x \in [-\frac{a}{2}, 0]$. Since $[-\frac{1}{b} + \frac{a}{2}, -\frac{1}{b} + a] \cup [-\frac{1}{b} + a, -\frac{1}{b} + \frac{3a}{2}] = [-\frac{1}{b} + \frac{a}{2}, -\frac{1}{b} + \frac{3a}{2}]$, this means precisely that

$$\Delta_a^2 g(x) \geq 0, \quad x \in [-\frac{1}{b} + \frac{a}{2}, -\frac{1}{b} + \frac{3a}{2}]. \quad (2.6)$$

We note that for $\frac{2}{N+a} < b \leq \frac{4}{N+3a}$,

$$-\frac{N}{2} \leq -\frac{1}{b} + \frac{a}{2}, \quad -\frac{1}{b} + \frac{3a}{2} \leq -\frac{N}{4} + \frac{3a}{4}.$$

Together with Lemma 2.2 this implies that (2.6) holds, as desired. \square

After this preparation we can now show that $G(x)$ is indeed invertible for $x \in [-\frac{a}{2}, \frac{a}{2}]$ under the assumptions in Theorem 1.2.

Corollary 2.4 *For $N > 0$, let $0 < a < N$ and $\frac{2}{N+a} < b \leq \frac{4}{N+3a}$. Assume that $g \in V_{N,a}$. Then $\det G(x) \neq 0$ for $x \in [-\frac{a}{2}, \frac{a}{2}]$.*

Proof. First, consider $x \in [-\frac{a}{2}, 0]$. By Lemma 2.3 (c), we have

$$\begin{aligned} \det G(x) &= -g(x-a)G_{21}(x) + g(x)G_{22}(x) - g(x+a)G_{23}(x) \\ &\geq -g(x-a)G_{21}(x) + g(x)(G_{21}(x) + G_{23}(x)) - g(x+a)G_{23}(x) \\ &= (g(x) - g(x-a))G_{21}(x) + (g(x) - g(x+a))G_{23}(x) \\ &=: A_N(x). \end{aligned}$$

Using Lemma 2.1 (c) and Lemma 2.3 (a) & (b),

$$(g(x) - g(x-a))G_{21}(x) \geq 0 \quad \text{and} \quad (g(x) - g(x+a))G_{23}(x) \geq 0.$$

Thus $A_N(x) \geq 0$, $x \in [-\frac{a}{2}, 0]$. If $A_N(x) > 0$ for all $x \in [-\frac{a}{2}, 0]$ the proof is completed; thus the rest of the proof will focus on the case where $A_N(x_0) = 0$ for some $x_0 \in [-\frac{a}{2}, 0]$. In this case Lemma 2.1 (c) shows that either

$$G_{21}(x_0) = 0, \quad G_{23}(x_0) = 0 \quad (2.7)$$

or

$$G_{21}(x_0) = 0, \quad x_0 = -\frac{a}{2}. \quad (2.8)$$

The case (2.8) actually can not occur. Indeed, if $G_{21}(-\frac{a}{2}) = 0$, then by Lemma 2.3 (a), we have $g(-\frac{a}{2} + \frac{1}{b}) = 0$; by the symmetry of g this would imply that $g(\frac{a}{2} - \frac{1}{b}) = 0$, which contradicts Lemma 2.1 (b) with $x = -\frac{a}{2}$. Thus we only have to deal with the case (2.7). If $G_{21}(x_0) = G_{23}(x_0) = 0$, then by Lemma 2.3 (a) & (b), we have

$$g(x_0 + \frac{1}{b}) = g(x_0 + \frac{1}{b} + a) = g(x_0 - \frac{1}{b} - a) = g(x_0 - \frac{1}{b}) = 0.$$

Inserting this information into the entries of the matrix $G(x_0)$ and applying Lemma 2.1 yields that

$$\det G(x_0) = g(x_0 + \frac{1}{b} - a)g(x_0)g(x_0 - \frac{1}{b} + a) > 0,$$

as desired. This completes the proof that $G(x) > 0$ for $x \in [-\frac{a}{2}, 0]$. Since g is symmetric around the origin, we have

$$\begin{aligned} \det G(-x) &= \det (g(-x - \frac{\ell}{b} + ma))_{-1 \leq \ell, m \leq 1} = \det (g(x + \frac{\ell}{b} - ma))_{-1 \leq \ell, m \leq 1} \\ &= -\det (g(x - \frac{\ell}{b} - ma))_{-1 \leq \ell, m \leq 1} = \det (g(x - \frac{\ell}{b} + ma))_{-1 \leq \ell, m \leq 1} \\ &= \det G(x). \end{aligned}$$

Thus $G(x)$ is also invertible for $x \in [0, \frac{a}{2}]$. \square

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2: By Corollary 2.4 and continuity of g , $\inf_{x \in [-\frac{a}{2}, \frac{a}{2}]} |\det G(x)| > 0$. We define h on $[-\frac{3a}{2}, \frac{3a}{2}]$ by

$$\begin{pmatrix} h(x - a) \\ h(x) \\ h(x + a) \end{pmatrix} = G^{-1}(x) \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \quad x \in [-\frac{a}{2}, \frac{a}{2}],$$

which is a bounded function. On $\mathbb{R} \setminus [-\frac{3a}{2}, \frac{3a}{2}]$, put $h(x) = 0$. It follows immediately by definition of h that then g and h are dual windows. \square

Remark 2.5 We note that the above approach is tailored to the region of parameters (a, b) in Theorem 1.2. For example, it does not apply to the region considered in Proposition 1.1 (ii). In fact, if $0 < b \leq \frac{2}{N+a}$, then the first row of $G(x)$ for $x \in [\frac{N}{2} - \frac{1}{b} + a, \frac{a}{2}]$ is the zero vector; and the third row of $G(x)$ for $x \in [-\frac{a}{2}, \frac{1}{b} - a - \frac{N}{2}]$ is the zero vector. Hence we have $\inf_{x \in [-\frac{a}{2}, \frac{a}{2}]} \det G(x) = 0$.

The conditions for $g \in V_{N,a}$ are technical. We will now give an example, showing that the conclusion about the size of the support of the dual window in Theorem 1.2 might break down if $g \notin V_{N,a}$.

Example 2.6 Let us consider the parameters $N = 5$, $a = 1$. Consider a symmetric and continuous function g on \mathbb{R} with $\text{supp } g = [-\frac{5}{2}, \frac{5}{2}]$; assume further that g is increasing on $[-\frac{N}{2}, 0] = [-\frac{5}{2}, 0]$, and that

$$g(0) = 12, \quad g(-1) = 10, \quad g(-\frac{4}{3}) = 5, \quad g(-\frac{7}{3}) = 3.$$

Then $\Delta_a^2 g(-\frac{4}{3}) = g(-\frac{4}{3}) - 2g(-\frac{7}{3}) + g(-\frac{10}{3}) = -1 < 0$. Hence g does not satisfy (A3) at $x = -\frac{4}{3}$, i.e. $g \notin V_{N,a}$.

We will show that for $b = \frac{3}{N+2a} = \frac{3}{7}$ there does not exist a bounded real-valued function $h \in L^2(\mathbb{R})$ with $\text{supp } h \subseteq [-\frac{3a}{2}, \frac{3a}{2}]$, such that the duality conditions (2.1) hold. In order to obtain a contradiction, let us assume that such a dual window indeed exists. Let $x_0 = 0$. Then

$$\begin{aligned} x_0 - a = -x_0 - a = -1, \quad x_0 - \frac{1}{b} + a = -x_0 - \frac{1}{b} + a = -\frac{4}{3}, \\ x_0 - \frac{1}{b} = -x_0 - \frac{1}{b} = -\frac{7}{3}, \quad x_0 - \frac{1}{b} - a = -x_0 - \frac{1}{b} - a = -\frac{10}{3}, \end{aligned}$$

and consequently

$$G(x_0) = \begin{pmatrix} g(x_0 + \frac{1}{b} - a) & g(x_0 + \frac{1}{b}) & g(x_0 + \frac{1}{b} + a) \\ g(x_0 - a) & g(x_0) & g(x_0 + a) \\ g(x_0 - \frac{1}{b} - a) & g(x_0 - \frac{1}{b}) & g(x_0 - \frac{1}{b} + a) \end{pmatrix} = \begin{pmatrix} 5 & 3 & 0 \\ 10 & 12 & 10 \\ 0 & 3 & 5 \end{pmatrix}.$$

By the continuity of g , there exist continuous functions $\epsilon_{ij}(x)$, $1 \leq i, j \leq 3$ such that

$$G(x) = \begin{pmatrix} 5 + \epsilon_{11}(x) & 3 + \epsilon_{12}(x) & \epsilon_{13}(x) \\ 10 + \epsilon_{21}(x) & 12 + \epsilon_{22}(x) & 10 + \epsilon_{23}(x) \\ \epsilon_{31}(x) & 3 + \epsilon_{32}(x) & 5 + \epsilon_{33}(x) \end{pmatrix}$$

and

$$\epsilon_{ij}(x) \rightarrow 0 \text{ as } x \rightarrow x_0.$$

Then (2.2) implies that

$$\begin{pmatrix} 5 + \epsilon_{11}(x) & 3 + \epsilon_{12}(x) & \epsilon_{13}(x) \\ 10 + \epsilon_{21}(x) & 12 + \epsilon_{22}(x) & 10 + \epsilon_{23}(x) \\ \epsilon_{31}(x) & 3 + \epsilon_{32}(x) & 5 + \epsilon_{33}(x) \end{pmatrix} \begin{pmatrix} h(x-a) \\ h(x) \\ h(x+a) \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$$

for a.e. $x \in [-\frac{a}{2}, \frac{a}{2}]$. By elementary row operations, this leads to

$$\begin{pmatrix} 5 + \epsilon_{11}(x) & 3 + \epsilon_{12}(x) & \epsilon_{13}(x) \\ \eta_1(x) & \eta_2(x) & \eta_3(x) \\ \epsilon_{31}(x) & 3 + \epsilon_{32}(x) & 5 + \epsilon_{33}(x) \end{pmatrix} \begin{pmatrix} h(x-a) \\ h(x) \\ h(x+a) \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \text{ a.e. } x \in [-\frac{a}{2}, \frac{a}{2}],$$

where $\eta_i(x) := \epsilon_{2i}(x) - 2\epsilon_{1i}(x) - 2\epsilon_{3i}(x)$ for $1 \leq i \leq 3$. Ignoring a possible set of measure zero and using that h is a bounded function, this implies that

$$b = \eta_1(x)h(x-a) + \eta_2(x)h(x) + \eta_3(x)h(x+a) \rightarrow 0$$

as $x \rightarrow x_0$. This is a contradiction. \square

On the other hand, the condition $g \in V_{N,a}$ is not a necessary condition for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame in the considered region. For example, let $N = 1$, $a = \frac{1}{4}$, $b = 2$ and take $g_1(x) := (e^{-x^2} - e^{-\frac{1}{4}})\chi_{[-\frac{1}{2}, \frac{1}{2}]}(x)$. Then elementary calculations show that (A3) does not hold for $x \in [-\frac{1}{10}, -\frac{1}{16}] \subset [-\frac{N}{2}, -\frac{N}{4} + \frac{3a}{4}]$. But since $\det G(x) > 0$ for $x \in [-\frac{a}{2}, \frac{a}{2}]$, one can prove that $\{E_{mb}T_{na}g_1\}_{m,n \in \mathbb{Z}}$ is a frame by following the steps in the proof of Theorem 1.2.

3 The set $V_{N,a}$

In this section we give easy verifiable sufficient conditions for a function g to belong to $V_{N,a}$. Recall that a continuous function $f : \mathbb{R} \rightarrow \mathbb{C}$ with $\text{supp } f = [-\frac{N}{2}, \frac{N}{2}]$ is piecewise continuously differentiable if there exist finitely many $x_0 = -\frac{N}{2} < x_1 < \dots < x_n = \frac{N}{2}$ such that

- (1) f is continuously differentiable on $] -x_{i-1}, x_i[$ for every $i \in \{1, \dots, n\}$;
- (2) the one-sided limits $\lim_{x \rightarrow x_{i-1}^+} f'(x)$ and $\lim_{x \rightarrow x_i^-} f'(x)$ exist for every $i \in \{1, \dots, n\}$.

Note that if g is a continuous and piecewise continuously differentiable function, the fundamental theorem of calculus yields that

$$\Delta_a g(x) = \int_{x-a}^x g'(t)dt \text{ for all } x \in \mathbb{R}, a > 0. \quad (3.1)$$

In order to avoid a tedious presentation, we will forego to mention the points where a piecewise continuously differentiable function is not differentiable, e.g., in conditions (c) and (d) in the following Proposition 3.1. The result is inspired by explicit calculations for B-splines, due to Trebels and Steidl; see Lemma 1 in [28].

Proposition 3.1 *Let $N > 0$ and assume that a continuous and piecewise continuously differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$ with $\text{supp } g = [-\frac{N}{2}, \frac{N}{2}]$ satisfies the following conditions:*

- (a) *g is symmetric around the origin;*
- (b) *g is strictly increasing on $[-\frac{N}{2}, 0]$;*
- (c) *g' is increasing on $]-\frac{N}{2}, -\frac{N}{4}]$;*
- (d) *$g'(-x - \frac{N}{2}) \leq g'(x)$ for $x \in [-\frac{N}{4}, 0[$.*

Then $g \in \cap_{0 < a < N} V_{N,a}$.

Proof. Note that the conditions (a) and (b) are exactly the same as (A1) and (A2). Thus we will prove (A3). In the entire argument we will assume that g is differentiable; an elementary consideration then extends the result to the case of piecewise differentiable functions. Let us first consider $x \leq -\frac{N}{4}$. Then by the mean value theorem

$$\Delta_a^2 g(x) = a \left(\frac{g(x) - g(x-a)}{a} - \frac{g(x-a) - g(x-2a)}{a} \right) = a(g'(\xi) - g'(\eta))$$

for some $\eta \in [x-2a, x-a]$, $\xi \in [x-a, x]$. Since g' is increasing up to $x = -\frac{N}{4}$ this proves that $\Delta_a^2 g(x) \geq 0$ whenever $x \leq -\frac{N}{4}$.

We now consider $x \in [-\frac{N}{4}, \min(-\frac{N}{4} + \frac{3a}{4}, 0)]$. Then

$$x - a \leq \min(-\frac{N}{4} - \frac{a}{4}, -a) \leq -\frac{N}{4}.$$

Then

$$\begin{aligned} \Delta_a^2 g(x) &= \int_{x-a}^{-\frac{N}{4}} (g'(t) - g'(t-a)) dt + \int_{-\frac{N}{4}}^x (g'(t) - g'(t-a)) dt \\ &\geq \int_{x-a}^{-\frac{N}{4}} (g'(t) - g'(t-a)) dt + \int_{-\frac{N}{4}}^x (g'(-t - \frac{N}{2}) - g'(t-a)) dt \\ &= g(-\frac{N}{4}) - g(-\frac{N}{4} - a) - g(x-a) + g(x-2a) \\ &\quad - g(-x - \frac{N}{2}) - g(x-a) + g(\frac{N}{4} - \frac{N}{2}) + g(-\frac{N}{4} - a) \\ &= g(-\frac{N}{4}) - g(x-a) - [g(-a - \frac{N}{4}) - g(x-2a)] \tag{3.2} \\ &\quad + g(-\frac{N}{4}) - g(x-a) - [g(-x - \frac{N}{2}) - g(-a - \frac{N}{4})]. \tag{3.3} \end{aligned}$$

We now consider the terms (3.2) and (3.3) separately. For (3.2), by the mean value theorem,

$$\begin{aligned}
& g(-\frac{N}{4}) - g(x-a) - [g(-a - \frac{N}{4}) - g(x-2a)] \\
&= (-\frac{N}{4} - x + a) \left(\frac{g(-\frac{N}{4}) - g(x-a)}{-\frac{N}{4} - x + a} - \frac{g(-a - \frac{N}{4}) - g(x-2a)}{-\frac{N}{4} - x + a} \right) \\
&= (-\frac{N}{4} - x + a) (g'(\xi) - g'(\eta))
\end{aligned}$$

for some $\eta \in [x-2a, -a - \frac{N}{4}]$, $\xi \in [x-a, -\frac{N}{4}]$. Since $-a - \frac{N}{4} \leq x-a$ we have $\eta \leq \xi \leq -\frac{N}{4}$; thus, by assumption (c), $g'(\eta) \leq g'(\xi)$. Recalling that $x \leq -\frac{N}{4} + \frac{3a}{4} < -\frac{N}{4} + a$ we have $-\frac{N}{4} - x + a > 0$; thus, we conclude that the term in (3.2) indeed is nonnegative.

For (3.3) we split into two cases. If $x-a \geq -x - \frac{N}{2}$, exactly the same argument as for (3.2) works. If $x-a < -x - \frac{N}{2}$ we perform the same argument after a rearrangement of the terms. Indeed,

$$\begin{aligned}
& g(-\frac{N}{4}) - g(x-a) - [g(-x - \frac{N}{2}) - g(-a - \frac{N}{4})] \\
&= g(-\frac{N}{4}) - g(-x - \frac{N}{2}) - [g(x-a) - g(-a - \frac{N}{4})] \\
&= (x + \frac{N}{4}) \left(\frac{g(-\frac{N}{4}) - g(-x - \frac{N}{2})}{x + \frac{N}{4}} - \frac{g(x-a) - g(-a - \frac{N}{4})}{x + \frac{N}{4}} \right) \\
&= (x + \frac{N}{4}) (g'(\xi) - g'(\eta))
\end{aligned}$$

for some $\eta \in [-a - \frac{N}{4}, x-a]$, $\xi \in [-x - \frac{N}{2}, -\frac{N}{4}]$. As before this implies that (3.3) is nonnegative.

We now consider $x = -\frac{N}{4} + \frac{3a}{4}$; according to (1.2) we must prove that

$$g(\frac{N}{4} - \frac{3a}{4}) - 2g(-\frac{N}{4} - \frac{a}{4}) \geq 0. \quad (3.4)$$

First, if $a > \frac{2N}{3}$ (i.e., $-\frac{N}{4} + \frac{3a}{4} > \frac{N}{4}$) we have

$$\begin{aligned}
& g(\frac{N}{4} - \frac{3a}{4}) - 2g(-\frac{N}{4} - \frac{a}{4}) \\
&= \frac{N-a}{2} \left(\frac{g(\frac{N}{4} - \frac{3a}{4}) - g(-\frac{N}{4} - \frac{a}{4})}{\frac{N-a}{2}} - \frac{g(-\frac{N}{4} - \frac{a}{4}) - g(-\frac{3N}{4} + \frac{a}{4})}{\frac{N-a}{2}} \right) \\
&= \frac{N-a}{2} (g'(\xi) - g'(\eta))
\end{aligned}$$

for some $\xi \in [-\frac{N}{4} - \frac{a}{4}, \frac{N}{4} - \frac{3a}{4}]$, $\eta \in [-\frac{3N}{4} + \frac{a}{4}, -\frac{N}{4} - \frac{a}{4}]$; thus (3.4) holds by assumption (c).

Now assume that $\frac{N}{2} < a \leq \frac{2N}{3}$; then $-\frac{N}{4} + \frac{3a}{4} \leq \frac{N}{4}$, and

$$-\frac{N}{4} \leq \frac{N}{4} - \frac{3a}{4} \leq 0, \quad -\frac{N}{4} - \frac{a}{4} < -\frac{N}{2} + \frac{a}{4} < 0.$$

Thus,

$$\begin{aligned} & g\left(\frac{N}{4} - \frac{3a}{4}\right) - 2g\left(-\frac{N}{4} - \frac{a}{4}\right) \\ & \geq g\left(-\frac{N}{4}\right) - g\left(-\frac{N}{4} - \frac{a}{4}\right) - (g\left(-\frac{N}{4} - \frac{a}{4}\right) - g\left(-\frac{N}{4} - \frac{2a}{4}\right)); \end{aligned}$$

this can again be expressed in terms of a difference $g'(\xi) - g'(\eta)$ with $\xi \in [-\frac{N}{4} - \frac{a}{4}, -\frac{N}{4}]$, $\eta \in [-\frac{N}{4} - \frac{2a}{4}, -\frac{N}{4} - \frac{a}{4}]$ and is hence positive.

Finally, we assume that $\frac{N}{3} \leq a \leq \frac{N}{2}$. Then $-\frac{N}{4} + \frac{3a}{4} \leq \frac{N}{4}$; since $-\frac{N}{4} - \frac{a}{4} < -\frac{a}{2}$, the assumption (b) implies that

$$g\left(\frac{N}{4} - \frac{3a}{4}\right) - g\left(-\frac{N}{4} - \frac{a}{4}\right) \geq g\left(\frac{N}{4} - \frac{3a}{4}\right) - g\left(-\frac{a}{2}\right) \geq \int_{-\frac{a}{2}}^{\frac{N}{4} - \frac{3a}{4}} g'(t) dt = (*).$$

Since $-\frac{N}{4} \leq -\frac{a}{2} < \frac{N}{4} - \frac{3a}{4} \leq 0$, (d) implies that

$$(*) \geq \int_{-\frac{a}{2}}^{\frac{N}{4} - \frac{3a}{4}} g'(-t - \frac{N}{2}) dt = \int_{-\frac{3N}{4} + \frac{3a}{4}}^{-\frac{N}{2} + \frac{a}{2}} g'(t) dt = (**).$$

Since $-\frac{N}{2} + \frac{a}{2} \leq -\frac{N}{4}$ and g' is increasing on $]-\infty, -\frac{N}{4}]$, shifting the integration interval by $-\frac{N}{4} + \frac{3a}{4} \geq 0$ to the left yields that

$$(**) \geq \int_{-\frac{N}{2}}^{-\frac{N}{4} - \frac{a}{4}} g'(t) dt = g\left(-\frac{N}{4} - \frac{a}{4}\right);$$

thus (3.4) holds, as desired. \square

Proposition 3.1 immediately leads to the following simple criterion for a function to belong to $\cap_{0 < a < N} V_{N,a}$.

Corollary 3.2 *Let $N > 0$, and assume that a continuous and piecewise continuously differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$ with $\text{supp } g = [-\frac{N}{2}, \frac{N}{2}]$ satisfies the following conditions:*

- (a) *g is symmetric around the origin;*

(b) g' is positive and increasing on $] -\frac{N}{2}, 0[$.

Then $g \in \bigcap_{0 < a < N} V_{N,a}$.

We will now describe several functions belonging to $V_{N,a}$, either for all $a \in]0, N[$ or a subinterval hereof.

Example 3.3 Consider the B-splines B_N , $N \in \mathbb{N}$, defined recursively by

$$B_1 = \chi_{[-1/2, 1/2]}, \quad B_{N+1} = B_N * B_1.$$

In [28, Lemma 1] it is proved that for $N \in \mathbb{N} \setminus \{1\}$,

- (i) B'_N is increasing on $] -\frac{N}{2}, -\frac{N}{4} + \frac{1}{4}]$;
- (ii) $B'_N(-x - \frac{N}{2}) \leq B'_N(x)$ for $x \in [-\frac{N}{4}, 0[$.

Thus Proposition 3.1 implies that $B_N \in \bigcap_{0 < a < N} V_{N,a}$ for $N \in \mathbb{N} \setminus \{1\}$. \square

Example 3.4 Let $N \in \mathbb{N} \setminus \{1\}$ and define

$$f_N(x) := \cos^{2N-2}\left(\frac{\pi x}{N}\right) \chi_{[-\frac{N}{2}, \frac{N}{2}]}(x).$$

Direct calculations show that for $x \in] -\frac{N}{2}, -\frac{N}{4}]$,

$$f''_N(x) = \frac{(2N-2)\pi^2}{N^2} \cos^{2N-4}\left(\frac{\pi x}{N}\right) \left((2N-3) \sin^2\left(\frac{\pi x}{N}\right) - \cos^2\left(\frac{\pi x}{N}\right) \right) \geq 0$$

and for $x \in [-\frac{N}{4}, 0[$,

$$f'_N(x) - f'_N(-x - \frac{N}{2}) = \frac{(2N-2)\pi}{2N} \sin\left(\frac{2\pi x}{N}\right) \left(\sin^{2N-4}\left(\frac{\pi x}{N}\right) - \cos^{2N-4}\left(\frac{\pi x}{N}\right) \right) \geq 0.$$

By Proposition 3.1, $f_N \in \bigcap_{0 < a < N} V_{N,a}$. \square

In the following examples we consider continuous and compactly supported “variants” of the two-sided exponential function, the Gaussian, and other classical functions.

Example 3.5 Let $N > 0$ and define

$$h_N(x) := \left(e^{-|x|} - e^{-\frac{N}{2}} \right) \chi_{[-\frac{N}{2}, \frac{N}{2}]}(x).$$

Then $h_N \in \bigcap_{0 < a < N} V_{N,a}$ by Corollary 3.2. \square

Example 3.6 Let $N > 0$ and define

$$k_N(x) := \left(\frac{1}{1+|x|} - \frac{1}{1+\frac{N}{2}} \right) \chi_{[-\frac{N}{2}, \frac{N}{2}]}(x).$$

Then $k_N \in \bigcap_{0 < a < N} V_{N,a}$ by Corollary 3.2. \square

In the following examples the simple sufficient conditions in Proposition 3.1 and Corollary 3.2 are not satisfied. We will use the definition directly to show that the considered functions belong to $V_{N,a}$ for certain ranges of the parameter a .

Example 3.7 Let $N > 0$, and consider

$$p_N(x) := \left(\frac{1}{1+x^2} - \frac{1}{1+(\frac{N}{2})^2} \right) \chi_{[-\frac{N}{2}, \frac{N}{2}]}(x).$$

We will show that

$$(a) \ p_N \in \bigcap_{\frac{3N}{7} \leq a < N} V_{N,a};$$

$$(b) \ p_N \in \bigcap_{\frac{N}{3} \leq a < \frac{3N}{7}} V_{N,a} \text{ if } N \geq \sqrt{\frac{12}{5}} \approx 1.5451 \dots$$

It is clear that (A1) and (A2) hold, so for the considered values for a we now check (A3). In fact the argument below will prove more, namely that

$$\Delta_a^2 p_N(x) \geq 0, \ x \in [-\frac{N}{2}, -\frac{N}{4} + \frac{3a}{4}]. \quad (3.5)$$

Considering any $a \in [\frac{N}{3}, N]$, we have

$$-\frac{N}{2} < -\frac{N}{4} + \frac{3a}{4} < \frac{N}{2}, \ a - \frac{N}{2} < -\frac{N}{4} + \frac{3a}{4} < a + \frac{N}{2}, \ -\frac{N}{4} + \frac{3a}{4} < 2a - \frac{N}{2};$$

so the fact that $p_N > 0$ on $] -\frac{N}{2}, \frac{N}{2}[$, $p_N(\cdot - a) > 0$ on $]a - \frac{N}{2}, a + \frac{N}{2}[$ and $p_N(\cdot - 2a) > 0$ on $]2a - \frac{N}{2}, 2a + \frac{N}{2}[$ immediately shows that

- (1) $p_N(x) \neq 0, \ x \in] -\frac{N}{2}, -\frac{N}{4} + \frac{3a}{4}];$
- (2) $p_N(x - a) = 0, \ x \in [-\frac{N}{2}, a - \frac{N}{2}]; \ p_N(x - a) \neq 0, \ x \in]a - \frac{N}{2}, -\frac{N}{4} + \frac{3a}{4}];$
- (3) $p_N(x - 2a) = 0, \ x \in [-\frac{N}{2}, -\frac{N}{4} + \frac{3a}{4}].$

Thus

$$\Delta_a^2 p_N(x) = \begin{cases} p_N(x), & x \in [-\frac{N}{2}, -\frac{N}{2} + a]; \\ p_N(x) - 2p_N(x-a), & x \in]-\frac{N}{2} + a, -\frac{N}{4} + \frac{3a}{4}]. \end{cases}$$

Since $p_N(x) \geq 0$, $x \in [-\frac{N}{2}, -\frac{N}{2} + a]$, it is now enough to prove that

$$p_N(x) - 2p_N(x-a) \geq 0, \quad x \in]-\frac{N}{2} + a, -\frac{N}{4} + \frac{3a}{4}]. \quad (3.6)$$

Let $x \in]-\frac{N}{2} + a, -\frac{N}{4} + \frac{3a}{4}]$. We see that

$$\begin{aligned} p_N(x) - 2p_N(x-a) &= p_N(x) - p_N(x-a) - p_N(x-a) \\ &= \frac{1}{1+x^2} - \frac{1}{1+(x-a)^2} - \left(\frac{1}{1+(x-a)^2} - \frac{1}{1+(\frac{N}{2})^2} \right) \\ &= \frac{(x-a)^2 - x^2}{(1+x^2)(1+(x-a)^2)} - \frac{(\frac{N}{2})^2 - (x-a)^2}{(1+(x-a)^2)(1+(\frac{N}{2})^2)}. \end{aligned} \quad (3.7)$$

In order to prove (a), we now assume that $\frac{3N}{7} \leq a < N$. Since $x^2 \leq (\frac{N}{2})^2$ and $(x-a)^2 - x^2 \geq 0$, we have

$$\begin{aligned} p_N(x) - 2p_N(x-a) &\geq \frac{(x-a)^2 - x^2}{(1+(\frac{N}{2})^2)(1+(x-a)^2)} - \frac{(\frac{N}{2})^2 - (x-a)^2}{(1+(x-a)^2)(1+(\frac{N}{2})^2)} \\ &= \frac{h(x)}{(1+(\frac{N}{2})^2)(1+(x-a)^2)}, \end{aligned}$$

where

$$h(x) := 2(x-a)^2 - x^2 - \frac{N^2}{4} = (x-2a)^2 - 2a^2 - \frac{N^2}{4}.$$

Note that the quadratic function h is symmetric around $x = 2a$. Since $-\frac{N}{4} + \frac{3a}{4} < 2a$ and $\frac{3N}{7} \leq a < N$, we have

$$h(x) \geq h(-\frac{N}{4} + \frac{3a}{4}) = \frac{1}{16}(N-a)(7a-3N) \geq 0. \quad (3.8)$$

Thus

$$p_N(x) - 2p_N(x-a) \geq 0.$$

Therefore (3.5) holds, i.e., the proof of (a) is completed.

In order to prove (b), assume now that $N \geq \sqrt{\frac{12}{5}}$ and $\frac{N}{3} \leq a < \frac{3N}{7}$. Since $(x-a)^2 - x^2 = 2a(\frac{a}{2} - x)$ and $(\frac{N}{2})^2 - (x-a)^2 = (\frac{N}{2} - x + a)(\frac{N}{2} + x - a)$, (3.7) implies that

$$p_N(x) - 2p_N(x-a) = \frac{1}{1+(x-a)^2} \left(\frac{2a(\frac{a}{2} - x)}{1+x^2} - \frac{(\frac{N}{2} - x + a)(\frac{N}{2} + x - a)}{1+(\frac{N}{2})^2} \right).$$

Using $-\frac{N}{2} + a < x \leq -\frac{N}{4} + \frac{3a}{4}$, it follows that $\frac{N}{2} - x + a < N$ and $\frac{N}{2} + x - a \leq \frac{N}{4} - \frac{a}{4} \leq -x + \frac{a}{2}$; thus

$$\begin{aligned} p_N(x) - 2p_N(x-a) &\geq \frac{1}{1+(x-a)^2} \left(\frac{2a(\frac{a}{2} - x)}{1+x^2} - \frac{N(\frac{a}{2} - x)}{1+(\frac{N}{2})^2} \right) \\ &= \frac{\frac{a}{2} - x}{1+(x-a)^2} \left(\frac{q(x)}{(1+x^2)(1+(\frac{N}{2})^2)} \right), \end{aligned}$$

where $q(x) := 2a(1+(\frac{N}{2})^2) - N(1+x^2)$. Since $\frac{N}{3} \leq a < \frac{3N}{7}$ and $-\frac{N}{2} + a < x \leq -\frac{N}{4} + \frac{3a}{4}$, we have $-\frac{N}{6} < x \leq \frac{N}{14}$; so $x^2 < \frac{N^2}{36}$. Thus

$$q(x) \geq 2\frac{N}{3} \left(1 + \left(\frac{N}{2} \right)^2 \right) - N \left(1 + \frac{N^2}{36} \right) = N \left(-\frac{1}{3} + \frac{5N^2}{36} \right) \geq 0.$$

Thus $p_N(x) - 2p_N(x-a) \geq 0$. Therefore (3.5) holds, as desired. \square

Example 3.8 Let $N > 0$, and consider

$$g_N(x) := \left(e^{-x^2} - e^{-\frac{N^2}{4}} \right) \chi_{[-\frac{N}{2}, \frac{N}{2}]}(x).$$

We will show that $g_N \in \bigcap_{\frac{3N}{7} \leq a < N} V_{N,a}$. As in Example 3.7, see (3.6), it suffices to check that

$$g_N(x) - 2g_N(x-a) \geq 0, \quad x \in]-\frac{N}{2} + a, -\frac{N}{4} + \frac{3a}{4}]$$

Let $x \in]-\frac{N}{2} + a, -\frac{N}{4} + \frac{3a}{4}]$, and $a \in [\frac{3N}{7}, N[$. We see that

$$\begin{aligned} g_N(x) - 2g_N(x-a) &= g_N(x) - g_N(x-a) - g_N(x-a) \\ &= e^{-x^2} - e^{-(x-a)^2} - \left(e^{-(x-a)^2} - e^{-\frac{N^2}{4}} \right) \\ &= e^{-(x-a)^2} \left(e^{(x-a)^2 - x^2} - 1 \right) - e^{-\frac{N^2}{4}} \left(e^{\frac{N^2}{4} - (x-a)^2} - 1 \right). \end{aligned}$$

Since $-(x-a)^2 \geq -\frac{N^2}{4}$ and $(x-a)^2 - x^2 \geq 0$, we have

$$\begin{aligned} g_N(x) - 2g_N(x-a) &\geq e^{-\frac{N^2}{4}} \left(e^{(x-a)^2 - x^2} - 1 \right) - e^{-\frac{N^2}{4}} \left(e^{\frac{N^2}{4} - (x-a)^2} - 1 \right) \\ &= e^{-\frac{N^2}{4}} \left(e^{(x-a)^2 - x^2} - e^{\frac{N^2}{4} - (x-a)^2} \right) \\ &= e^{-(x-a)^2} \left(e^{h(x)} - 1 \right), \end{aligned}$$

where $h(x) := 2(x-a)^2 - x^2 - (\frac{N}{2})^2$. From (3.8), we have $h(x) \geq 0$. Thus

$$g_N(x) - 2g_N(x-a) \geq 0,$$

as desired. \square

Competing interests: The authors declare that they have no competing interests.

Authors' contributions: All authors contributed equally to this work. All authors read and approved the final manuscript.

Acknowledgments: The authors would like to thank the reviewers for many useful suggestions, which clearly improved the presentation. In particular, one reviewer suggested to formulate the condition (A3) for membership of $V_{N,a}$ in terms of second order differences, which is much more transparent than our original condition. He also suggested the approach in the current proof of Proposition 3.1, which is shorter than our original proof. This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2013R1A1A2A10011922).

References

- [1] Ascensi, G, Feichtinger, HG, Kaiblinger, N: Dilation of the Weyl symbol and the Balian-Low theorem. Trans. Amer. Math. Soc. **366**, 3865–3880 (2014)
- [2] Christensen, O: An introduction to frames and Riesz bases. Second expanded edition. Birkhäuser, Boston (2016)
- [3] Christensen, O, Kim, HO, Kim, RY: Gabor windows supported on $[-1, 1]$ and compactly supported dual windows. Appl. Comput. Harmon. Anal. **28**, 89–103 (2010)

- [4] Christensen, O, Kim, HO, Kim, RY: Gabor windows supported on $[-1, 1]$ and dual windows with small support. *Adv. Comput. Math.* **36**, 525–545 (2012)
- [5] Christensen, O, Kim, HO, Kim, RY: On entire functions restricted to intervals, partition of unities, and dual Gabor frames. *Appl. Comput. Harmon. Anal.* **38**, 72–86 (2015)
- [6] Christensen, O, Kim, HO, Kim, RY: On Gabor frames generated by sign-changing windows and B-splines. *Appl. Comput. Harmon. Anal.* **39**, 534–544 (2015)
- [7] Dai, XR, Sun, Q: The abc-problem for Gabor systems. *Mem. Amer. Math. Soc.* to appear (2015)
- [8] Del Prete, V: Estimates, decay properties, and computation of the dual function for Gabor frames. *J. Fourier Anal. Appl.* **5**, 545–562 (1999)
- [9] Feichtinger, HG, Kaiblinger, N: Varying the time-frequency lattice of Gabor frames. *Trans. Amer. Math. Soc.* **356**, 2001–2023 (2004)
- [10] Gröchenig, K: Partitions of unity and new obstructions for Gabor frames, preprint.
- [11] Gröchenig, K, Janssen, AJEM, Kaiblinger, N, Pfander, G: Note on B-splines, wavelet scaling functions, and Gabor frames. *IEEE Trans. Inform. Theory.* **49**, 3318–3320 (2003)
- [12] Gröchenig, K, Stöckler, J: Gabor frames and totally positive functions. *Duke Math. J.* **162**, 1003–1031 (2013)
- [13] Janssen, AJEM: Some Weyl-Heisenberg frame bound calculations. *Indag. Math.* **7**, 165–183 (1996)
- [14] Janssen, AJEM: The duality condition for Weyl-Heisenberg frames. In: Feichtinger, HG, Strohmer, T (eds.) *Gabor analysis: theory and application*, pp. 33–84. Birkhäuser, Boston (1998)
- [15] Janssen, AJEM: Zak transforms with few zeros and the tie. In: Feichtinger, HG, Strohmer, T (eds.) *Advances in Gabor analysis*, pp. 31–70. *Appl. Numer. Harmon. Anal.* Birkhäuser, Boston (2003)
- [16] Janssen, AJEM: On generating tight Gabor frames at critical density. *J. Fourier Anal. Appl.* **9**, 175–214 (2003)

- [17] Janssen, AJEM, Strohmer, T: Hyperbolic secants yield Gabor frames. Appl. Comput. Harmon. Anal. **12**, 259–267 (2002)
- [18] Kim, I: Gabor Frames with trigonometric spline dual windows. Asian-Eur. J. Math. **8**(4), 1550072 (2015)
- [19] Kloos, T, Stöckler, J: Zak transforms and Gabor frames of totally positive functions and exponential B-splines. J. Approx. Theory. **184**, 209–237 (2014)
- [20] Laugesen, RS: Gabor dual spline windows. Appl. Comput. Harmon. Anal. **27**, 180–194 (2009)
- [21] Lemvig, J, Nielsen, H: Counterexamples to the B-spline conjecture for Gabor frames. J. Fourier Anal. Appl., to appear.
- [22] Lyubarskii, Y: Frames in the Bargmann space of entire functions. Adv. Soviet Math. **11**, 167–180 (1992)
- [23] Lyubarskii, Y, Nes, PG: Gabor frames with rational density. Appl. Comput. Harmon. Anal. **34**, 488–494 (2013)
- [24] Prete, VD: Estimates, decay properties, and computation of the dual function for Gabor frames. J. Fourier Anal. Appl. **5**, 545–561 (1999)
- [25] Ron, A, Shen, Z: Weyl-Heisenberg systems and Riesz bases in $L^2(\mathbb{R}^d)$. Duke Math. J. **89**, 237–282 (1997)
- [26] Seip, K: Density theorems for sampling and interpolation in the Bargmann-Fock space I. J. Reine Angew. Math. **429**, 91–106 (1992)
- [27] Seip, K, Wallsten, R: Density theorems for sampling and interpolation in the Bargmann-Fock space II. J. Reine Angew. Math. **429**, 107–113 (1992)
- [28] Trebels, B, Steidl, G: Riesz bounds of Wilson bases generated by B-splines. J. Fourier Anal. Appl. **6**, 171–184 (2000)